5.4 Indefinite Integrals and the Net Change Theorem

In this section we will focus on how to evaluate and understand the indefinite integral. The notation $\int f(x)dx$ is used for an antiderivative of $\boldsymbol{f}$ and is called the indefinite integral.

$$\int f(x)dx = F(x) \quad \textit{means} \quad F'(x) = f(x)$$

We understand this to be true by using the relationship between antiderivatives and integrals given by the Fundamental Theorem of Calculus.

NOTE: It is very important that you distinguish the differences between the **definite** and **indefinite** integral. A definite integral, $\int_a^b f(x)dx$, is a number, whereas an indefinite integral, $\int f(x)dx$, is a function (or family of functions).

Below we have a table of derivatives and their inverse antiderivatives. These are also found in the back reference pages of your Calculus text.

Integration Rules:

1. $\int f(x)dx = F(x) + C \Leftrightarrow F'(x) = f(x)$
2. $\int a\, f(x)dx = a \int f(x)dx$
3. $\int -f(x)dx = -\int f(x)dx$
4. $\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\, dx = x + C$
2. $\int a\, dx = ax + C$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\, dx = -\cos x + C$
5. $\int \cos x\, dx = \sin x + C$
6. $\int \sec^2 x\, dx = \tan x + C$
7. $\int \csc^2 x\, dx = -\cot x + C$
8. $\int \sec x(\tan x)\, dx = \sec x + C$
9. $\int \csc x(\cot x)\, dx = -\csc x + C$
10. $\int \frac{1}{x} dx = \ln|x| + C$
11. $\int e^x dx = e^x + C$
12. $\int a^x dx = \frac{a^x}{\ln a} + C\ \ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C$
16. $\int \tan x\, dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$
17. $\int \cot x\, dx = \ln|\sin x| + C$ or $-\ln|\csc x| + C$
18. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
19. $\int \csc x\, dx = \ln|\csc x - \cot x| + C$
20. $\int \ln x\, dx = x\ln|x| - x + C$
21. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$ or $\frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$
24. $\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$. Note: $\sin^2 x = \frac{1-\cos 2x}{2}$

You must know the derivative rules in order to remember the antiderivative/integration rules.

Recall that the most general antiderivative on a general interval is obtained by adding a constant to a antiderivative. We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.

For example, $\int \frac{1}{x^2}dx = -\frac{1}{x} + c$ is written with the understanding that it is valid on the interval $(-\infty, 0) \cup (0, \infty)$. (Notice – this is the domain of the antiderivative.)

Example: Find $\int \left(\cos x + \frac{1}{2}x\right) dx$

$$\int \left(\cos x + \frac{1}{2}x\right) dx = \int \cos x \, dx + \int \frac{1}{2}x \, dx = \int \cos x \, dx + \frac{1}{2}\int x \, dx = \sin x + \frac{1}{2}\left(\frac{x^2}{2}\right) + c = \mathbf{\sin(x)} + \frac{\mathbf{x^2}}{\mathbf{4}} + \mathbf{c}$$

You can check your solution by differentiating it to see if you get the function in the integral.

Example: Find $\int x(\sqrt[3]{x} + \sqrt[4]{x})dx$ First rewrite $\sqrt[3]{x}$ and $\sqrt[4]{x}$ as powers – then multiply.

$$\int x(x^{\frac{1}{3}} + x^{\frac{1}{4}})dx = \int \left(x \cdot x^{\frac{1}{3}} + x \cdot x^{\frac{1}{4}}\right) dx = \int \left(x^{\frac{4}{3}} + x^{\frac{5}{4}}\right) dx$$

$$\frac{x^{\frac{7}{3}}}{\frac{7}{3}} + \frac{x^{\frac{9}{4}}}{\frac{9}{4}} + C = \underline{\frac{\mathbf{3}x^{\frac{7}{3}}}{\mathbf{7}} + \frac{\mathbf{4}x^{\frac{9}{4}}}{\mathbf{9}} + \boldsymbol{C} = \frac{\mathbf{3}}{\mathbf{7}}\sqrt[3]{\boldsymbol{x}^7} + \frac{\mathbf{4}}{\mathbf{9}}\sqrt[4]{\boldsymbol{x}^9} + \boldsymbol{C} = \frac{\mathbf{3}}{\mathbf{7}}\boldsymbol{x}^2\sqrt[3]{\boldsymbol{x}} + \frac{\mathbf{4}}{\mathbf{9}}\boldsymbol{x}^2\sqrt[4]{\boldsymbol{x}} + \boldsymbol{C}}$$

(Any of the underlined, bold solutions are acceptable.)

APPLICATIONS:

Part 2 of the FTC says that if ***f*** is continuous and [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where **F** is any antiderivative of *f*. This means that **F ' =** ***f***, so the equation can be rewritten as

$$\int_a^b F'(x)dx = F(b) - F(a).$$

We know that ***F'(x)*** represents the rate of change of ***y* = *F(x)*** with respect to ***x*** and **F(b) – F(a)** is the change in ***y*** when ***x*** changes from **a** to **b.**

Net Change Theroem: The integral of a rate of change is the ***net change***.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If ***V(t)*** is the velocity of water in a reservoir at time ***t***, then its derivative ***V'(t)*** is the rate at which water flows into a reservoir at time ***t***. So

$$\int_{t_1}^{t_2} V'(t)dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If the mass of a rod measured form the left end to a point x is $m(x)$, then the linear density is $p(x) = m'(x)$. So

$$\int_a^b p(x)dx = m(b) - m(a).$$

is the mass of the segent of the rod that lies between $x = a$ to $x = b$.

- The acceleration of an object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)dt = v(t_2) + v(t_1)$$

is the change in velocity from time t_1 to time t_2.

There are other applications in the Calculus text.

Example: The velocity of a jogger (mph) is $v(t) = 2t^2 - 8t + 6$, for $0 \leq t \leq 3$ where t is measured in hours.

a) Find the displacement (in miles) over the interval [0, 1].

Use the fact that if the position function is $s(t)$, then its velocity is $V(t) = s'(t)$, so

$$\int_{t_1}^{t_2} V(t)dt = s(t_2) - s(t_1) \quad or$$

$$s(1) - s(0) = \int_0^1 (2t^2 - 8t + 6)dt$$

$$= \frac{2t^3}{3} - \frac{8t^2}{2} + 6t\Big]_0^1$$

$$= \left(\frac{2(1)^3}{3} - \frac{8(1)^2}{2} + 6(1)\right) - \left(\frac{2(0)^3}{3} - \frac{8(0)^2}{2} + 6(0)\right)$$

$$= \frac{2}{3} - 4 + 6 = \frac{2 - 12 + 18}{3} = \frac{8}{3}$$